Reasoning about Structural Properties of Social Networks

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Overview

• Reasoning about structural properties of networks in modal logic
• The logic of social networks: brief overview of existing approaches
• Structural balance
• A minimal modal logic for reasoning about balance
  • Completeness

Reasoning about structural properties in modal logic: idea

Atomic propositions $p$: basic properties of agents ("is Dutch", "likes bacon")

Complex formulas $\phi$: complex properties of agents situated in a social network ("only has female friends", "is linked by no more than seven steps to Kevin Bacon")

Frames: social networks

Models: social networks annotated with basic properties
Examples: PDL

(family)killer
does the agent (e.g., the victim) have a family relation to the killer?

(family)male
are all politicians the agent has a family relation to male?

(neighbour; family)president
does the agent have a neighbour who is in family with the president?

(neighbour ∪ family)president
is the agent either a neighbour of, or in family with, the president?

(friend*)killer
is the agent (e.g., the victim) indirectly related to the killer through a chain of friends?

Examples: PDL (cont.)

((works_with ∪ family)*)gonzales
is there an indirect connection, via work and family relationships, between the agent (e.g., Richard Nixon) and Gonzales (one of the Watergate burglars)? (I.e., is there a chain of agents from Nixon to Gonzales where each agent in the chain is related to the next through either work or family?)

Examples: CTL

E(politician ∧ convicted)
the agent is directly related to a politician with a conviction.

E◊killer
the agent (e.g., the victim) is indirectly related to the killer. I.e., there is a chain of agents $a_1, \ldots, a_n$ such that $a_i$ is related to $a_{i+1}$ and $a_1$ is the victim (the current agent) and $a_n$ the killer.

A◊(politician → male)
all politicians the agent is related to are males.

A □(politician → E◊journalist)
all politicians the agent is indirectly related to, are indirectly related to a journalist.

Examples: CTL (cont.)

E(freemason ∪ private_investigator)
the agent is indirectly connected to a private investigator, through a chain of agents whom all are freemasons.
Examples: multi-dimensional CTL (Ågotnes et al., 2009)

$E_{\text{family}} \circ \text{killer}$

is the agent (e.g., the victim) directly related to the killer via the family relation?

$E_{\text{neighbour}} \circ E_{\text{family}} \circ \text{president}$

does the agent have a neighbour in family with the president?

Examples: ALT (turn-based synchronous)

$\neg ((\{\text{ann, bill}\}) \Diamond \neg \text{policeman})$

no matter who ann and bill choose as their friend, respectively, there will be a friend-of-a-friend path from the current agent to a police man (where ann and bill are restricted to their chosen friends). This might for example be a relevant query if one is interested in information flow: no matter whom of their friends ann and bill confide in, their secret can reach a police man.

As a query language

(form* killer

Input: model and formula

Model checker

Output: "no" and a counterexample (sequence of states where the property does not hold)

Formalising reasoning about networks: existing approaches
Epistemic approaches

- Model the often subtle consequences of events in social networks on higher-order knowledge and belief

\[ n \ F K (F) n \]
(all my friends know they are friends with me)

Approaches formalising concepts from social network analysis

- Information cascades: (Christoff, 2016; Baltag et al., 2013), dynamic epistemic logic
- Diffusion: (Christoff, 2016; Christoff and Hansen, 2015), dynamic hybrid logic
- Opinion dynamics: (Hansen, 2015), extended fuzzy logic

Approaches to reasoning about network structure

- Logical models of social networks:
  1. Network structure: binary social relation over a set of agents
  2. Features of agents: atomic propositions
- Dynamics
  - Change in the network structure based on features (typically similarity) (Smets and Velazquez-Quesada, LORI 2017, DALI 2017)
  - Change in the features based on network structure (Baltag et al., Studia Logica 2018)
- Epistemics are (again!) also often also taken into account.

Existing approaches

- Mainly focussed on information flow, less on structural properties
- Do not consider the nature of relations: weak or strong, friend or enemy, ...
- Don’t assume anything about the structure of the network (apart from basic properties like symmetry of the friendship relation)
Reasoning about Balance

Positive and Negative Relationships

- Relationships can be of many different types
- One common distinction: positive vs. negative
  - typically “friends” or “enemies”
  - modelled using signed graphs

Complete graphs

- Assume first that the social network graph is complete
  - Everyone knows (friend or enemy) everyone else

Structural balance

- Key idea: for any group of three people, some combinations of + and - are more plausible than others in real social networks
  - Unbalanced triangles occur less frequently in social settings
+/- combinations in groups of threes

Balanced

Unbalanced

Structural balance

- A (complete) graph is structurally balanced iff every group of three nodes is balanced.

The balance theorem

any structurally balanced (complete) graph can be divided into a group X of mutual friends, and a group Y of mutual friends, such that everyone in X is an enemy of everyone in Y.

General balance: possible definitions

Local: possible to fill in “missing” edges to get a balanced complete network

Global: possible to divide the network into to mutually opposed sets with only positive internal edges
Balance: general definition

The following are equivalent:
- it is possible to fill in “missing” edges to get a balanced complete network (local view)
- it is possible to divide the network into to mutually opposed sets with only positive internal edges (global view)
- there is no cycle with an odd number of negative edges

n-balance

- Full balance: ideal state that the network converges towards if nothing else changes.
- Unrealistic on large networks (only a single bad cycle needed for unbalance!)
- Long cycles with an odd number of negative edges are more likely to occur than short ones
  - "It well documented fact that the longer cycles have less effect upon a person’s tension than the shorter ones" (Estrada and Benzi, 2014)
- Definitions of balance that are used in practice often only take the shortest cycles into account, or give less weight to longer than to shorter cycles
- This motivates our definition of n-balance

Def. n-balance: let n be a natural number. A (general) signed social network is n-balanced iff it has no cycles of length less than or equal to n with an odd number of negative edges.
- An approximation of full balance, taking only the shortest cycles into account
- A measure of the degree of balance (the highest number n such that the network is n-balanced)

Equivalent: any sub-graph with n nodes is balanced.
A minimal modal logic

Language and semantics

\[ \phi ::= p \mid \neg \phi \mid \phi \lor \phi \mid \Diamond \phi \mid \Box \phi \]

\[ \Box \phi \equiv \neg \Diamond \neg \phi \]

**Definition (Models).** A model is a tuple \((A, R^+, R^-, V)\) consisting of

- A set \(A\) of agents
- Binary relations \(R^+\) and \(R^-\) on \(A\)
- A valuation function \(V : \text{PROP} \to \mathcal{P}(A)\)

such that

- \(R^+\) is reflexive: you are your own friend
- \(R^+ \cap R^- = \emptyset\): someone can’t be both your friend and your enemy

A model is \(n\)-balanced iff the graph \((A, R^+, R^-)\) is.

Interpretation (standard!)

Formulas are interpreted as properties of agents.

\[ M, a \models p \quad \text{iff} \quad a \in V(p), \]
\[ M, a \models \neg \phi \quad \text{iff} \quad M, a \not\models \phi, \]
\[ M, a \models \phi_1 \lor \phi_2 \quad \text{iff} \quad M, a \models \phi_1 \text{ or } M, a \models \phi_2, \]
\[ M, a \models \Diamond \phi \quad \text{iff} \quad \text{there exists } b \in A, a R^+ b \text{ and } M, b \models \phi, \]
\[ M, a \models \Box \phi \quad \text{iff} \quad \text{there exists } b \in A, a R^- b \text{ and } M, b \models \phi \]

\[ \models \phi: M, a \models \phi \text{ for all } M \text{ and } a \]

\[ \models_n \phi: M, a \models \phi \text{ for all } n\text{-balanced } M \text{ and } a \]

Logical properties

Undefinability of non-overlapping (reflexive loops omitted): \(F_1\) (left) and \(F_2\) (right).

- non-overlapping
- \(n\)-balance
- not modally definable
Strategy: find suitable sufficient conditions for \( n \)-balance

Find a set of formulas \( S = \bigcup_{\chi \in \mathcal{C}} S_{\chi} \) such that:

1. any \( \phi \in S \) is sufficient for \( n \)-balance: \( M, a \models \phi \Rightarrow \text{there is no odd cycle of length } \leq n \) starting in \( a \)
2. the following rule is sound

\[
\vdash \phi \to \chi \quad \phi \in S_{\chi}
\]

(2) ensures that any finite consistent set of formulas can be extended to a finite consistent set containing a formula in \( S \).

(1) ensures \( n \)-balance if a formula in \( S \) holds in every state.

\[\begin{align*}
\text{Suitable sufficient conditions} \\
\text{name}_{n}(\phi, \psi) = \\
\Box (\phi \land \neg \psi) \land \bigwedge_{n-1} (\Box; \Box) O_{\neg \phi \lor \psi} \land \Box (\neg \phi \land \psi) \land \bigwedge_{n-1} (\Box; \Box) E_{\neg \phi \lor \neg \psi}
\end{align*}\]

Lemma. For any \( \phi, \psi, n, M, a \):

\[M, a \models \text{name}_{n}(\phi, \psi)\]

\[\Downarrow\]

\( M \) has no cycle of length \( \leq n \) starting in \( a \)

Lemma. The rule \( \text{Nb}_{n} \)

\[\vdash \text{name}_{n}(p, q) \to \chi \quad \text{where } p, q \notin P(\chi) \text{ and } p \neq q\]

\[\vdash \chi\]

is sound

\[\begin{align*}
\text{Axiomatic system (parameterised by } n) \\
\text{Let:} \\
\bullet (\Box; \Box)^{x, y}_{\phi} \text{ be the set of all formulas that are obtained by prefixing } \phi \text{ with a sequence of } x \text{ positive (} \Box \text{) and } y \text{ negative (} \Box \text{) box modalities in some order;} \\
\bullet \bigwedge_{n} (\Box; \Box)^{O, a}_{\phi} \text{ be the conjunction of all } \bigwedge_{n} (\Box; \Box)^{E, c}_{\phi} \text{ such that } x + a = n \text{ and } a \text{ is an odd number;} \\
\bullet \bigwedge_{n} (\Box; \Box)^{E, c}_{\phi} \text{ be the conjunction of all } \bigwedge_{n} (\Box; \Box)^{E, c}_{\phi} \text{ such that } x + c = n \text{ and } c \text{ is an even number.}
\end{align*}\]

\[\text{name}_{n}(\phi, \psi) = \Box(\phi \land \neg \psi) \land \bigwedge_{n-1} (\Box; \Box)^{O, a}_{\neg \phi \lor \psi} \land \Box (\neg \phi \land \psi) \land \bigwedge_{n-1} (\Box; \Box)^{E, c}_{\neg \phi \lor \neg \psi}\]

\[\begin{align*}
\text{Figure 1: Axiomatization } \text{pul}_{n}, \text{ where } p, q \in \text{PROP}, \text{ and } L \in \{\Box, \Box\} \text{ and } M \in \\
(\Box, \Box) \text{ are the respective dual operators.}
\end{align*}\]

\[\begin{align*}
\text{Theorem.} \text{ For any natural number } n, \text{ the system is sound and complete with respect to all } n \text{-balanced models.}
\end{align*}\]
Outline of completeness proof

- Define standard canonical “model”. Undefinability n-balance => will not be a proper model
- Use the step-by-step method to define a submodel of the canonical “model” that:
  - is a proper model
  - is n-balanced (ensured by including a name-formula in each state, which the Nb rule allows us to do)
  - we can prove a truth lemma for

Further results

**Theorem.** $\text{pul}_1$ is sound and complete with respect to the class of all models.

\[(4B) \quad ((\Diamond \lozenge p \lor \lozenge \Diamond p) \rightarrow \Diamond p) \land ((\Diamond \lozenge p \lor \lozenge \Diamond p) \rightarrow \lozenge p)\]

**Theorem.** $\text{pul}_1 + 4B$ is sound and complete with respect to the class of all balanced complete models.

Summary

- Reasoning about structural properties of social networks
- Asked: what are the logical consequences of networks being balanced to a certain degree?
- Defined the notion of n-balance
- Studied family of minimal modal logics => fundamental principles of reasoning about balance
- Completeness results:
  - wrt. all networks
  - wrt. n-balanced networks, for any n
  - wrt. balanced complete networks

Future work

- Combine with epistemic social network logics such as (Seligman et al., 2013)
  - **Weak balance** (Davis, 1967)
  - **Full balance** on general (not necessarily complete) networks
  - Logical dynamics of the kinds of change caused by unbalance in the spirit of dynamic epistemic logic