Reasoning about Structural Properties of Social Networks

REINS closing conference, 11 February 2019

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Overview

- Reasoning about structural properties of networks in modal logic
- The logic of social networks: brief overview of existing approaches
- Structural balance
- A minimal modal logic for reasoning about balance
  - Completeness

Reasoning about structural properties in modal logics: examples

Reasoning about structural properties of social networks in modal logic: idea

Atomic propositions p: basic properties of *agents* ("is Dutch", "likes bacon")

Complex formulas ϕ: complex properties of agents situated in a social network ("only has female friends", "is linked by no more than seven steps to Kevin Bacon")

Frames: social networks

Models: social networks annotated with basic properties

Examples: PDL

$\langle family \rangle killer$
does the agent (e.g., the victim) have a family relation to the killer?

$[family] male$
are all politicians the agent has a family relation to male?

$\langle neighbour; family \rangle president$
does the agent have a neighbour who is in family with the president?

$\langle neighbour \cup family \rangle president$
is the agent either a neighbour of, or in family with, the president?

$\langle friend* \rangle killer$
is the agent (e.g., the victim) indirectly related to the killer through a chain of friends?

Examples: PDL (cont.)

$\langle (works\_with \cup family)* \rangle gonzales$
is there an indirect connection, via work and family relationships, between the agent (e.g., Richard Nixon) and Gonzales (one of the Watergate burglars)? (I.e., is there a chain of agents from Nixon to Gonzales where each agent in the chain is related to the next through either work or family?)

Examples: CTL

$\mathsf{E}\bigcirc(politician \wedge convicted)$
the agent is directly related to a politician with a conviction.

$\mathsf{E}\Diamond killer$
the agent (e.g., the victim) is *indirectly related to* the killer. I.e., there is a chain of agents $a_1, \ldots, a_n$ such that a_i is related to a_{i+1} and a_1 is the victim (the current agent) and a_n the killer.

$\mathsf{A}\bigcirc(politician \rightarrow male)$
all politicians the agent is related to are males.

$\mathsf{A}\Box(politician \rightarrow \mathsf{E}\Diamond journalist)$
all politicians the agent is indirectly related to, are indirectly related to a journalist.

Examples: CTL (cont.)

$\mathsf{E}(freemason\,\mathcal{U}\,private\_investigator)$
the agent is indirectly connected to a private investigator, through a chain of agents whom all are freemasons.

Examples: multi-dimensional CTL (Ågotnes et al., 2009)

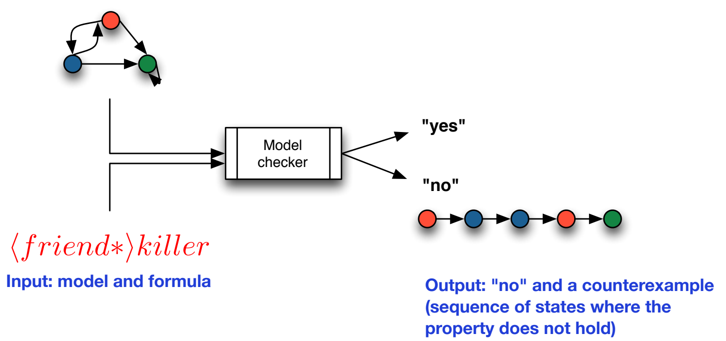
$\mathsf{E}_{family} \bigcirc killer$
is the agent (e.g., the victim) directly related to the killer via the family relation?

$\mathsf{E}_{neighbour} \bigcirc \mathsf{E}_{family} \bigcirc president$
does the agent have a neighbour in family with the president?

Examples: ALT (turn-based synchronous)

$\neg\langle\langle\{ann, bill\}\rangle\rangle\Diamond\neg policeman$
no matter who *ann* and *bill* choose as their friend, respectively, there will be a friend-of-a-friend path from the current agent to a police man (where *ann* and *bill* are restricted to their chosen friends). This might for example be a relevant query if one is interested in information flow: no matter whom of their friends *ann* and *bill* confide in, their secret can reach a police man.

As a query language



Formalising reasoning about networks: existing approaches

Epistemic approaches

- Model the often subtle consequences of events in social networks on higher-order knowledge and belief
- Pacuit and Parikh 2005; Wang et al 2010; Ruan and Thielscher 2011; Van Eijck and Sietsma 2011; Seligman et al 2011, 2013; van Ditmarsch et al 2016; Xiong et al 2017

$$\downarrow n\ FK\langle F\rangle n$$

(all my friends know they are friends with me)

Approaches formalising concepts from social network analysis

- Information cascades: (Christoff, 2016; Baltag et al., 2013), dynamic epistemic logic
- Diffusion: (Christoff, 2016; Christoff and Hansen, 2015), dynamic hybrid logic
- Opinion dynamics: (Hansen, 2015), extended fuzzy logic

Approaches to reasoning about network structure

- Logical models of social networks:
  1. Network structure: binary social relation over a set of agents
  2. Features of agents: atomic propositions
- Dynamics
  - Change in the network structure based on features (typically *similarity*) (Smets and Velazquez-Quesada, LORI 2017, DALI 2017) $[\odot_\theta]\phi$
  - Change in the features based on network structure (Baltag et al., Studia Logica 2018) $[adopt]\phi$
- *Epistemics* are (again!) also often also taken into account.

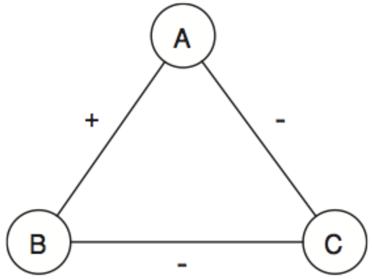
Existing approaches

- Mainly focussed on information flow, less on structural properties
- Do not consider the nature of relations: weak or strong, friend or enemy, …
- Don't assume anything about the structure of the network (apart from basic properties like symmetry of the friendship relation)

Reasoning about Balance

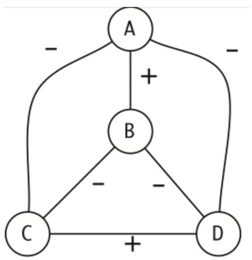
Positive and Negative Relationships

- Relationships can be of many different types
- One common distinction: positive vs. negative
  - typically “friends” or “enemies”
  - modelled using *signed* graphs



Complete graphs

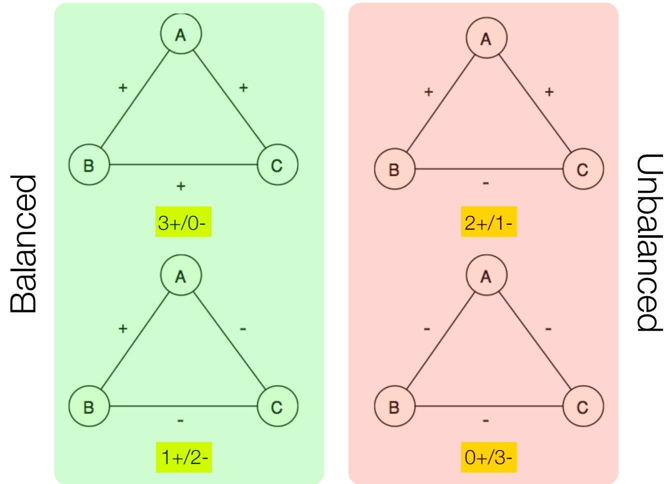
- Assume first that the social network graph is *complete*
  - Everyone knows (friend or enemy) everyone else



Structural balance

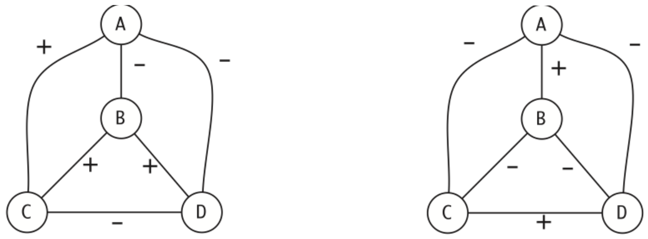
- Key idea: for any group of three people, some combinations of + and - are more plausible than others in real social networks
- *Unbalanced triangles* occur less frequently in social settings

+/- combinations in groups of threes



Structural balance

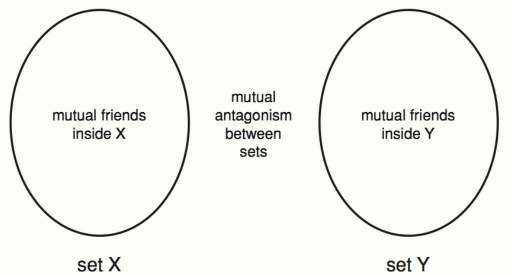
- A (complete) graph is structurally balanced iff every group of three nodes is balanced.



Not balanced

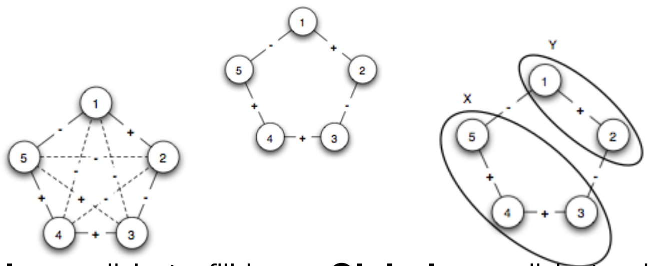
Balanced

The balance theorem



The balance theorem: *any* structurally balanced (complete) graph can be divided into a group X of mutual friends, and a group Y of mutual friends, such that everyone in X is an enemy of everyone in Y

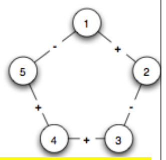
General balance: possible definitions



Local: possible to fill in "missing" edges to get a balanced complete network

Global: possible to divide the network into to mutually opposed sets with only positive internal edges

Balance: general definition



The following are equivalent:

- it is possible to fill in "missing" edges to get a balanced complete network (**local** view)
- it is possible to divide the network into to mutually opposed sets with only positive internal edges (**global** view)
- *there is no cycle with an odd number of negative edges*

n-balance

n-balance

- Full balance: *ideal state* that the network converges towards if nothing else changes.
  - Unrealistic on large networks (only a single bad cycle needed for unbalance!)
- *Long* cycles with an odd number of negative edges are more likely to occur than short ones
  - "*It well documented fact that the longer cycles have less effect upon a person's tension than the shorter ones*" (Estrada and Benzi, 2014)
- Definitions of balance that are used in practice often only take the shortest cycles into account, or give less weight to longer than to shorter cycles
- This motivates our definition of *n-balance*

n-balance

Def. n-balance: let n be a natural number. A (general) signed social network is *n-balanced* iff it has no cycles of length less than or equal to n with an odd number of negative edges.

- An *approximation* of full balance, taking only the shortest cycles into account
- A measure of the *degree* of balance (the highest number n such that the network is n-balanced)

Equivalent: any sub-graph with n nodes is balanced.

A minimal modal logic

Language and semantics

$\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \Diamond^{+} \phi \mid \Diamond^{-} \phi$

$\boxplus\phi \equiv \neg\Diamond^{+}\neg\phi$

$\boxminus\phi \equiv \neg\Diamond^{-}\neg\phi$

Definition (Models)**.** *A model is a tuple (A, R^+, R^-, V) consisting of*

- *A set A of* agents
- *Binary relations R^+ and R^- on A*
- *A valuation function V :* PROP $\to \mathcal{P}(A)$

such that

- *R^+ is reflexive: you are your own friend*
- *$R^+ \cap R^- = \emptyset$: someone can't be both your friend and your enemy*

A model is *n-balanced* iff the graph (A, R^+, R^-) is.

Interpretation (standard!)

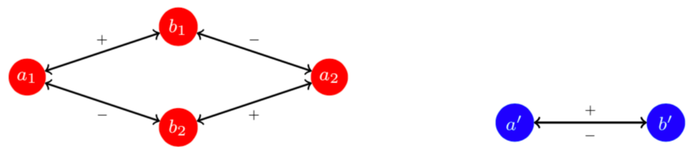
Formulas are interepreted as properties of agents.

$M, a \models p$	iff	$a \in V(p)$,
$M, a \models \neg\phi$	iff	$M, a \not\models \phi$,
$M, a \models \phi_1 \vee \phi_2$	iff	$M, a \models \phi_1$ or $M, a \models \phi_2$,
$M, a \models \Diamond^{+} \phi$	iff	there exists $b \in A, aR^+b$ and $M, b \models \phi$,
$M, a \models \Diamond^{-} \phi$	iff	there exists $b \in A, aR^-b$ and $M, b \models \phi$

$\models \phi$: $M, a \models \phi$ for all M and a

$\models_n \phi$: $M, a \models \phi$ for all n-balanced M and a

Logical properties



Undefinability of non-overlapping (reflexive loops omitted): $\mathcal{F}_1$ (left) and $\mathcal{F}_2$ (right).

non-overlapping	not modally definable
n-balance	not modally definable

Strategy: find suitable *sufficient* conditions for n-balance

Find a set of formulas $S = \bigcup_{\chi \in \mathcal{L}} S_\chi$ such that:

1. any $\phi \in S$ is *sufficient* for n-balance: $M, a \models \phi \Rightarrow$ there is no odd cycle of length $\leq n$ starting in a
2. the following rule is *sound*

$$\frac{\vdash \phi \to \chi}{\vdash \chi} \qquad \phi \in S_\chi$$

(2) ensures that any finite consistent set of formulas can be extended to a finite consistent set containing a formula in S.

(1) ensures n-balance if a formula in S holds in every state.

Suitable sufficient conditions

Let:

- $(\boxplus;\boxminus)^{x,y}_{\phi}$ be the set of all formulas that are obtained by prefixing ϕ with a sequence of x positive ($\boxplus$) and y negative ($\boxminus$) box modalities in some order;
- $\bigwedge_n (\boxplus;\boxminus)^{O}_{\phi}$ be the conjunction of all $\bigwedge (\boxplus;\boxminus)^{x,o}_{\phi}$ such that $x + o = n$ and o is an *odd* number;
- $\bigwedge_n (\boxplus;\boxminus)^{E}_{\phi}$ be the conjunction of all $\bigwedge (\boxplus;\boxminus)^{x,e}_{\phi}$ such that $x + e = n$ and e is an *even* number.

$$\mathit{name}_n(\phi,\psi) = \boxplus(\phi \wedge \neg\psi) \wedge \bigwedge_{n-1} (\boxplus;\boxminus)^{O}_{\neg\phi\vee\psi} \wedge \boxminus(\neg\phi \wedge \psi) \wedge \bigwedge_{n-1} (\boxplus;\boxminus)^{E}_{\phi\vee\neg\psi}$$

Suitable sufficient conditions

$$\mathit{name}_n(\phi,\psi) = \boxplus(\phi \wedge \neg\psi) \wedge \bigwedge_{n-1} (\boxplus;\boxminus)^{O}_{\neg\phi\vee\psi} \wedge \boxminus(\neg\phi \wedge \psi) \wedge \bigwedge_{n-1} (\boxplus;\boxminus)^{E}_{\phi\vee\neg\psi}$$

Lemma. *For any ϕ, ψ, n, M, a:*

$$M, a \models \mathit{name}_n(\phi,\psi)$$
$$\Downarrow$$
M has no cycle of length $\leq n$ starting in a

Lemma. *The rule* $\mathbf{Nb}_n$

$$\frac{\vdash \mathit{name}_n(p,q) \to \chi}{\vdash \chi} \qquad \text{where } p, q \notin P(\chi) \text{ and } p \neq q$$

is sound

Axiomatic system (parameterised by n)

(PC)	all substitution instances of propositional tautologies	
(T^{+})	$\vdash p \to \Diamond^{+} p$	(Positive-reflexivity)
$(B^{\pm})$	$\vdash p \to (\boxplus\Diamond^{+} p \wedge \boxminus\Diamond^{-} p)$	(Symmetry)
$(Dual)$	$\vdash Lp \leftrightarrow \neg M \neg p$	(Duality)
(K_s)	$\vdash M(p \to q) \to (Mp \to Mq)$	(Signed K-axiom)
(**MP**)	$\vdash \phi \to \psi \ \& \ \vdash \phi \Rightarrow \vdash \psi$	(Modus Ponens)
(**Nec**)	$\vdash \phi \Rightarrow \vdash M\phi$	(Signed Necessitation)
(**US**)	$\vdash \phi \Rightarrow \vdash \phi(\psi_1/p_1, \cdots, \psi_n/p_n)$	(Universal Substitution)
($\mathbf{Nb}_n$)	$\vdash \mathit{name}_n(p,q) \to \chi \ \Rightarrow \vdash \chi$, where $p, q \in \textsc{prop} \setminus P(\chi), p \neq q$	(n-balanced)

Figure 1: Axiomatization $\mathbf{pnl}_n$, where $p, q \in \textsc{prop}$, and $L \in \{\Diamond^{+}, \Diamond^{-}\}$ and $M \in \{\boxplus, \boxminus\}$ are the respective dual operators.

Theorem. *For any natural number n, the system is sound and complete with respect to all n-balanced models.*

Outline of completeness proof

- Define standard canonical “model”. Undefinability n-balance => will not be a proper model
- Use the step-by-step method to define a submodel of the canonical “model” that:
  - is a proper model
  - is n-balanced (ensured by including a name-formula in each state, which the **Nb** rule allows us to do)
  - we can prove a truth lemma for

Further results

Theorem. $\mathbf{pnl}_1$ *is sound and complete with respect to the class of all models.*

$$(4\text{B}) \quad ((\Diamond^{+}\Diamond^{+}p \vee \Diamond^{-}\Diamond^{-}p) \rightarrow \Diamond^{+}p) \wedge ((\Diamond^{+}\Diamond^{-}p \vee \Diamond^{-}\Diamond^{+}p) \rightarrow \Diamond^{-}p)$$

Theorem. $\mathbf{pnl}_1+4\text{B}$ *is sound and complete with respect to the class of all balanced complete models.*

Summary

- Reasoning about structural properties of social networks
- Asked: what are the logical consequences of networks being balanced to a certain degree?
- Defined the notion of n-balance
- Studied family of minimal modal logics => fundamental principles of reasoning about balance
- Completeness results:
  - wrt. all networks
  - wrt. n-balanced networks, for any n
  - wrt. balanced complete networks

Future work

- Combine with epistemic social network logics such as (Seligman et al., 2013)
- *Weak* balance (Davis, 1967)
- Full balance on general (not necessarily complete) networks
- Logical dynamics of the kinds of change caused by unbalance in the spirit of dynamic epistemic logic